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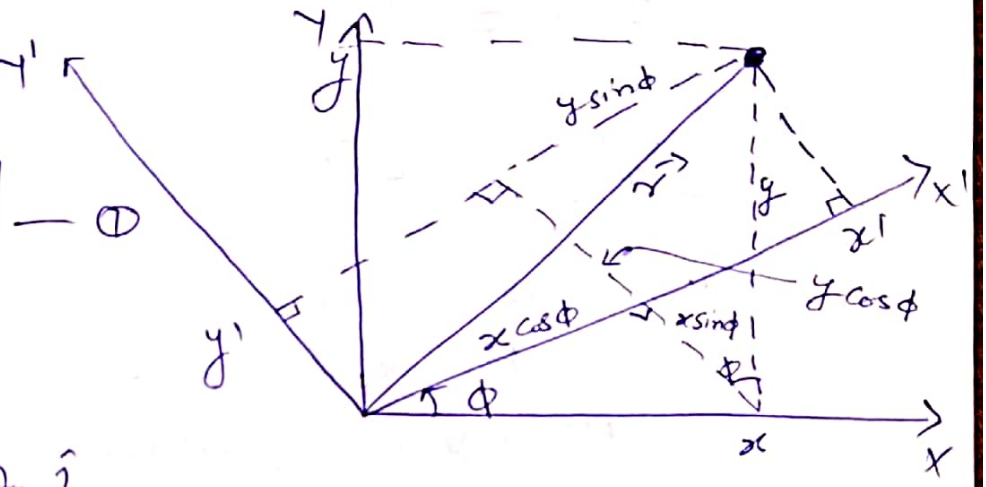
Mathematical Physics

B.Sc Physics. Part-III

① Rotation of Coordinate axes in 2D

$\vec{r} = x\hat{i} + y\hat{j}$



$$\left.\begin{aligned} x' &= x\cos\phi + y\sin\phi \\ y' &= -x\sin\phi + y\cos\phi \end{aligned}\right] \text{— } ①$$

Take vector $\vec{A}$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$

In rotated coordinate system $\vec{A'} = A'_x \hat{i} + A'_y \hat{j}$

$$\left.\begin{aligned} A'_x &= A_x\cos\phi + A_y\sin\phi \\ A'_y &= -A_x\sin\phi + A_y\cos\phi \end{aligned}\right] \text{— } ②$$

From ② $\quad A'^2_x + A'^2_y = A_x^2 + A_y^2$

$\Rightarrow |\vec{A}| = |\vec{A}'|$ { Scalar is invariant

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{— } ③$$

{ To get $\begin{bmatrix} x \\ y \end{bmatrix}$ find inverse of matrix

Now $x \to x_1$, $y \to x_2$ | $a_{11} = \cos\phi$ $\quad a_{12} = \sin\phi$, $a_{21} = -\sin\phi$, $a_{22} = \cos\phi$

We can write ③ as

$$x'_i = \sum_{j=1}^{2} a_{ij} x_j \quad ; \quad i = 1, 2 \text{ —— } ④$$

$a_{ij} \to$ direction cosine.

$a_{ij} \equiv \cos(x'_i, x_j)$

$a_{11} = \cos(x'_1, x_1) = \cos\phi$

$a_{12} = \cos(x'_1, x_2) = \cos(\phi - \pi/2) = \sin\phi$

$a_{21} = \cos(x'_2, x_1) = \cos(y', x) = \cos(\frac{\pi}{2} + \phi) = -\sin\phi$

Now generalized ④ to N dim

$$V_i' = \sum_{j=1}^{N} a_{ij} \cdot V_j \quad , \; i = 1, 2, \ldots, N \quad \text{—— (5)}$$

$$\begin{bmatrix} V_1' \\ V_2' \\ \vdots \\ V_N' \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & a_{2N} \\ \vdots & & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix}_{N \times N} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix}$$

$a_{ij} \rightarrow$ Cosine of angle between the x_i' direction and positive x_j direction.

We may write (cartesian coordinates)

$$a_{ij} = \frac{\partial x_i'}{\partial x_j'}$$

using inverse rotation $[\phi \rightarrow -\phi]$

$$x_j = \sum_{i=1}^{2} a_{ij} \cdot x_i' \implies \frac{\partial x_j'}{\partial x_i'} = a_{ij}$$

Now $$V_i' = \sum_{j=1}^{N} \frac{\partial x_i'}{\partial x_j} V_j = \sum_{j=1}^{N} \frac{\partial x_j}{\partial x_i'} V_j \quad \text{—— (6)}$$

The direction cosine a_{ij} satisfy (orthogonality condition)

$$\left.\begin{aligned} &\sum_i a_{ij} a_{ik} = \delta_{jk} \\ \text{or } &\sum_i a_{ji} a_{ki} = \delta_{jk} \end{aligned}\right] \text{—— (7)}$$

$$\delta_{jj} = 1 \, , \text{ for } j = k$$

$$\delta_{jk} = 0 \, , \text{ for } j \neq k$$

Gradient

Total variation of a scalar point function $\phi(x,y,z)$

$$d\phi(x,y,z) \equiv [\phi(x+dx, y+dy, z+dz) - \phi(x, y+dy, z+dz)]$$
$$+ [\phi(x, y+dy, z+dz) - \phi(x, y, z+dz)]$$
$$+ [\phi(x, y, z+dz) - \phi(x, y, z)]$$

$$d\phi(x,y,z) = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$\because$ $dF(x,y) = F(x+dx, y+dy) - F(x,y)$

$$= [F(x+dx, y+dy) - F(x, y+dy)] + [F(x, y+dy) - F(x,y)]$$

$$= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

adding and substracting this

Since ϕ is scalar

$$\phi'(x_1', x_2', x_3') = \phi(x_1, x_2, x_3)$$

$$\frac{\partial\phi'(x_1', x_2', x_3')}{\partial x_i'} = \frac{\partial\phi(x_1, x_2, x_3)}{\partial x_i'} = \sum_j \frac{\partial\phi}{\partial x_j'}\frac{\partial x_j'}{\partial x_i'} = \sum_j a_{ij}' \frac{\partial\phi}{\partial x_j'}$$

by eqn (6) we have constructed a vector with component $\frac{\partial\phi}{\partial x_j}$. This vector we label the gradient of ϕ

$$\nabla\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

$$\nabla = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

vector differential operator

Importance
$\vec{F} = -\vec{\nabla}U$

(2) Directional derivative; Gradient ( )

$T(x,y,z) \rightarrow$ Temperature

Let we know T at every point of a room or a metal b

$\Rightarrow$ rate of change of T with distance changes
(ϕ^s or t_s in some direction)

$\rightarrow$ Rate of change of T depends on direction in which we move; consequently it is called directional derivati

$\Rightarrow$ We want to find $\frac{\Delta T}{\Delta S}$ ← element of distance (arc length) in corresponding distance

We write directional derivative as $\frac{dT}{ds}$

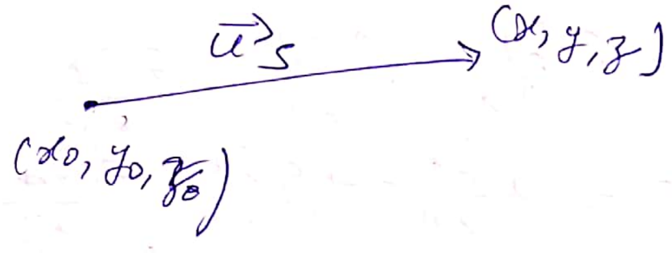
$\rightarrow$ heat flow from large $\frac{dT}{ds}$ to low $\frac{dT}{ds}$

Take scalar function $\phi(x,y,z)$

We want to calculate $\frac{d\phi}{ds}$

$\rightarrow$ rate of change of ϕ with distance at a given point (x_0, y_0, z_0) and in a given direction

$\rightarrow$ let $\vec{u} = ia + jb + kc$ $\rightarrow$ unit vector in given directi



We start from (x_0, y_0, z_0) and go a distance 's' in direction of $\vec{u}$ to point (x, y, z)

$\Downarrow$

Vector joining these two point is $\vec{u}s$.

{$\because \vec{u}$ is unit vector}

$$\therefore (x, y, z) - (x_0, y_0, z_0) = \vec{u}s = (a\hat{i} + b\hat{j} + c\hat{k})s$$

$$\Rightarrow \left.\begin{array}{l} x = x_0 + as \\ y = y_0 + bs \\ z = z_0 + cs \end{array}\right\} \quad \text{——} \quad (1)$$

$x, y, z \rightarrow$ functions of s only (single variable)

Now ϕ becomes function of one variables.

Now $$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds} \quad \text{—} \quad (2)$$

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x} a + \frac{\partial \phi}{\partial y} b + \frac{\partial \phi}{\partial z} c \quad \text{——} \quad (3)$$

We see that (3) is dot product of $\vec{u}$ with vector $i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}$

We call this vector as grad ϕ or $\nabla\phi$

$$\nabla\phi = \text{grad}\,\phi = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z} \quad \text{——} \quad (4)$$

$$\frac{d\phi}{ds} = \nabla\phi \cdot \vec{u} \longrightarrow \text{directional derivative}$$

(5)